

# Computation over Mismatched Channels

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## Abstract

We consider the problem of distributed computation of a target function over a two-user deterministic multiple-access channel. If the target and channel functions are matched (i.e., compute the same function), significant performance gains can be obtained by jointly designing the communication and computation tasks. However, in most situations there is mismatch between these two functions. In this work, we analyze the impact of this mismatch on the performance gains achievable with joint communication and computation designs over separation-based designs. We show that for most pairs of target and channel functions there is no such gain, and separation of communication and computation is optimal.

## I. INTRODUCTION

The problem of computing a function from distributed information arises in many different contexts ranging from auctions and financial trading to sensor networks. In order to compute the desired target function, communication between the distributed users is required. If this communication takes place over a shared medium, such as in a wireless setting, the channel introduces interactions between the transmitted signals. This suggests the possibility to harness these signal interactions to facilitate the task of computing the desired target function. A fundamental question is therefore whether by jointly designing encoders and decoders for communication and computation, we can improve the efficiency of distributed computation.

### A. Summary of Results

In this paper, we explore this question by considering computation of a function over a two-user multiple-access channel (MAC). In order to focus on the impact of the structural mismatch between the target and channel functions on the efficiency of computation, we ignore channel noise and consider only *deterministic* MACs here. More formally, the setting consists of two transmitters observing a (random) variable  $u_1 \in \mathcal{U}$  and  $u_2 \in \mathcal{U}$ , respectively, and a receiver aiming to compute the function  $a(u_1, u_2) \in \mathcal{W}$  of these variables. The two transmitters are connected to the destination through a deterministic MAC with inputs  $x_1, x_2 \in \mathcal{X}$  and output  $y = g(x_1, x_2) \in \mathcal{Y}$ , where  $g(\cdot, \cdot)$  describes the actions of the channel.

A straightforward achievable scheme for this problem is to separate the tasks of communication and computation: the transmitters communicate the values of  $u_1$  and  $u_2$  to the destination, which then uses these values to compute the desired target function  $a(u_1, u_2)$ . This requires the receiver to decode  $2 \log |\mathcal{U}|$  message bits. However, the MAC itself also computes a function  $g(x_1, x_2)$  of the two inputs  $x_1, x_2$ , creating the opportunity of taking advantage of the structure of  $g(\cdot, \cdot)$  to calculate  $a(\cdot, \cdot)$ . This is trivially possible when  $g(\cdot, \cdot)$  and  $a(\cdot, \cdot)$  are *matched*, i.e., compute the same function on their inputs. In such cases, performing the tasks of communication and computation jointly results in significantly fewer bits to be communicated. Indeed, in the matched case only the  $\log |\mathcal{W}|$  bits describing the function value are

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recovered at the receiver. This could be considerably less than the  $2 \log |\mathcal{U}|$  bits resulting from the separation approach. Naturally, in most cases the channel  $g(\cdot, \cdot)$  and the target function  $a(\cdot, \cdot)$  are *mismatched*. The question is thus whether we can still obtain performance gains over separation in this mismatched situation. In other words, we ask if in general the natural computation done by the channel can be harnessed to help with the computation of the desired target function.

We consider two cases: i) *One-shot communication*, where the MAC is used only once, but the channel input alphabet  $\mathcal{X}$  and output alphabet  $\mathcal{Y}$  are allowed to vary as a function of the domain  $\mathcal{U}$  of the target function. In this case, performance is measured in terms of the scaling needed for the channel alphabets with respect to the computation alphabets, i.e., how  $|\mathcal{X}|, |\mathcal{Y}|$  grow with  $|\mathcal{U}|$ . This is closer to the formulation in the computer science literature. ii) *Multi-shot communication*, where the channel alphabets  $|\mathcal{X}|, |\mathcal{Y}|$  are of fixed size, but the channel can be used several times. In this case, performance is measured in terms of computation rate, i.e., how many channel uses are needed to compute the target function. This is closer to the formulation considered in information theory.

As the main result of this paper, we show that separation between computation and communication is essentially optimal for most<sup>1</sup> pairs  $(a, g)$  of target and channel functions. In other words, the structural mismatch between the functions  $a(\cdot, \cdot)$  and  $g(\cdot, \cdot)$  is in general too strong for joint computation and communication designs to yield any performance gains.

We illustrate this with an example for one-shot communication. Assume that the variables  $u_1, u_2$  at the transmitters take on a large range of values, say  $|\mathcal{U}| = 2^{1000}$ , and the receiver is only interested in knowing if  $u_1 \geq u_2$ , i.e., in a binary target function. Then for most MACs and one-shot communication, a consequence<sup>2</sup> of Theorems 1 and 2 in Section III (illustrated in Example 3) is that the transmitters need to convey the *entire* values of  $u_1, u_2$  to the destination, which then simply compares them. Thus, even though the destination is interested in only a *single* bit about  $(u_1, u_2)$ , it is still necessary to transmit  $2 \log |\mathcal{U}| = 2000$  bits over the channel.

More generally, Theorems 1 and 2 in Section III together demonstrate that for most target functions separation of communication and computation is asymptotically optimal for most MACs. Example 4 illustrates that only for special functions like an equality check (i.e., checking whether  $u_1 = u_2$ ) can we significantly improve upon the simple separation scheme. Intuitively, this is because the structural mismatch between most target and channel functions is too large to allow for any possibility of direct computation of the target function value without resorting to recovering the user messages first. The technical ideas that enable these observations are based on a connection with results in extremal graph theory such as existence of complete subgraphs and matchings of a given size in a bipartite graph. These connections might be of independent interest.

Similarly, for multi-shot communication, where we repeatedly use a fixed channel, Theorem 4 in Section III shows that for most functions, the computation rate is necessarily as small as that for the identity target function describing the entire variables  $u_1, u_2$  at the destination. In other words, separation of communication and computation is again optimal for most target and channel functions. To prove this result, the usual approach using cut-set bound arguments is not tight enough. Indeed, Example 5 shows that the ratio between the upper bound on the computation rate obtained from the cut-set bound and the correct scaling derived in Theorem 4 can be unbounded. Rather, the structures of the target and channel functions have to be analyzed jointly.

These results show that, in general, there is little or no benefit in joint designs: computation-communication separation is optimal for most cases. We thus advocate in this paper that separation of computation and communication for multiple-access channels is not just an attractive option from an implementation point of view, but, except for special cases, actually entails little loss in efficiency.

<sup>1</sup>More precisely, among all target functions  $a(\cdot, \cdot)$  with given domain  $\mathcal{U}$  and range  $\mathcal{W}$ , and all channel functions  $g(\cdot, \cdot)$  with given input alphabet  $\mathcal{X}$  and output alphabet  $\mathcal{Y}$ , separation is optimal except for at most an exponentially small (in domain size  $|\mathcal{U}|$ ) fraction of pairs.

<sup>2</sup>While the theorems only present results in the limit as  $|\mathcal{U}| \rightarrow \infty$ , it follows from the proofs that for a given domain  $\mathcal{U}$  the statements hold for all but an exponentially small (in  $|\mathcal{U}|$ ) fraction of channel functions.

## B. Related Work

The problem of distributed function computation has a rich history and has been studied in many different contexts. In computer science, it has been studied under the branch of communication complexity, for example see [1] and references therein. Early seminal work by Yao [2] considered interactive communication between two parties. Among several other important results, the paper showed that the number of exchanged bits required to compute most target functions is as large as for the identity function. In the context of information theory, distributed function computation has been studied as an extension of distributed source coding in [3]–[5]. For example, Körner and Marton [3] showed that for the computation of the finite-field sum of correlated sources linear codes can outperform random codes. This was extended to large networks represented as graphs in [6]–[8] and references therein. Randomized gossip algorithms [9] have been proposed as practical schemes for information dissemination in large unreliable networks and were studied in the context of distributed computation in [9], [10] among several others.

In most of these works, communication channels are represented as orthogonal point-to-point links. When the channel itself introduces signal interaction, as is the case for a MAC, there can be a benefit from jointly handling the communication and computation tasks as illustrated in [11]. Function computation over MACs has been studied in [12]–[15] and references therein.

There is some work touching on the aspect of structural mismatch between the target and the channel functions. In [16], an example was given in which the mismatch between a linear target function with integer coefficients and a linear channel function with real coefficients can significantly reduce efficiency. In [15], it was conjectured that, for computation of finite-field addition over a real-addition channel, there could be a gap between the cut-set bound and the computation rate. In [17], mismatched computation when the network performs linear finite-field operations was studied. To the best of our knowledge, a systematic study of channel and computation mismatch is initiated in this work.

## C. Organization

The paper is organized as follows. In Section II, we formally introduce the questions studied in this paper. We present the main results along with illustrative examples in Section III. Most of the proofs are given in Section IV.

## II. PROBLEM SETTING AND NOTATION

Throughout this paper, we use sans-serif font for random variables, e.g.,  $u$ . We use bold font lower and upper case to denote vectors and matrices, e.g.,  $\mathbf{y}$  and  $\mathbf{G}$ . All sets are typeset in calligraphic font, e.g.,  $\mathcal{X}$ . We denote by  $\log(\cdot)$  and  $\ln(\cdot)$  the logarithms to the base 2 and  $e$ , respectively.

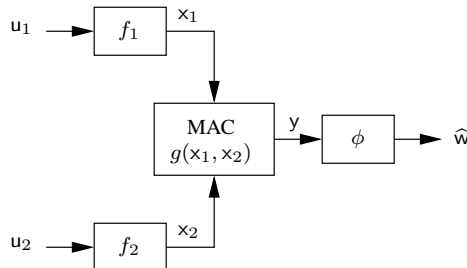


Fig. 1. Computation over a deterministic multiple-access channel. Each user  $i$  has access to an independent message  $u_i$ , and the receiver computes an estimate  $\hat{w}$  of the target function  $a(u_1, u_2)$  of those messages.

A discrete, memoryless, deterministic two-user MAC consists of two *input alphabets*  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , an *output alphabet*  $\mathcal{Y}$ , and a deterministic *channel function*  $g: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$ . Given channel inputs  $x_1, x_2$ , the output of the MAC is

$$y \triangleq g(x_1, x_2).$$

Each transmitter  $i \in \{1, 2\}$  has access to an independent and uniformly distributed *message*  $u_i \in \mathcal{U}_i$ . The objective of the receiver is to compute a *target function*  $a: \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathcal{W}$  of the user messages, see Fig. 1.

Formally, each transmitter  $i$  consists of an *encoder*  $f_i: \mathcal{U}_i \rightarrow \mathcal{X}_i$  mapping the message  $u_i$  into the channel input

$$x_i \triangleq f_i(u_i).$$

The receiver consists of a *decoder*  $\phi: \mathcal{Y} \rightarrow \mathcal{U}$  mapping the channel output  $y$  into an *estimate*

$$\hat{w} \triangleq \phi(y)$$

of the target function  $a(u_1, u_2)$ . The *probability of error* is

$$\mathbb{P}(a(u_1, u_2) \neq \phi(y)).$$

*Remark:* We point out that this differs from the ordinary communication setting, in which the decoder aims to recover both messages  $(u_1, u_2)$ . Instead, in the setting here, the decoder is not interested in  $(u_1, u_2)$ , but only in the value  $a(u_1, u_2)$  of the target function.

In the following, it will often be convenient to represent the target function  $a(\cdot, \cdot)$  and the channel  $g(\cdot, \cdot)$  by their corresponding matrices  $\mathbf{A} = (a_{u_1, u_2}) \in \mathcal{W}^{\mathcal{U}_1 \times \mathcal{U}_2}$  and  $\mathbf{G} = (g_{x_1, x_2}) \in \mathcal{Y}^{\mathcal{X}_1 \times \mathcal{X}_2}$ , respectively. In other words,

$$\begin{aligned} a_{u_1, u_2} &= a(u_1, u_2) \in \mathcal{W}, \\ g_{x_1, x_2} &= g(x_1, x_2) \in \mathcal{Y}. \end{aligned}$$

For  $n \in \mathbb{N}$ , denote by  $\mathbf{G}^{\otimes n}$  the  $n$ -fold use of the same channel matrix  $\mathbf{G}$ . In other words, the matrix  $\mathbf{G}^{\otimes n}$  describes the actions of the (memoryless) channel  $\mathbf{G}$  on the sequence

$$((x_1[1], x_2[1]), (x_1[2], x_2[2]), \dots, (x_1[n], x_2[n]))$$

of length  $n$  of channel inputs.

**Definition.** A pair  $(\mathbf{A}, \mathbf{G})$  of target and channel functions is  $\delta$ -feasible, if there exist encoders  $f_1, f_2$  and a decoder  $\phi$  computing the target function  $\mathbf{A}$  over  $\mathbf{G}$  with probability of error at most  $\delta$ .

*Remark:* We will often consider pairs  $(\mathbf{A}, \mathbf{G}^{\otimes n})$ , in which case the definition of  $\delta$ -feasibility allows for coding over  $n$  uses of the channel  $\mathbf{G}$ .

Without loss of generality, we assume that the target function  $\mathbf{A}$  has no two identical rows or two identical columns, since we could otherwise simply eliminate one of them. For ease of exposition, we will focus on the case

$$\begin{aligned} \mathcal{U}_1 &= \mathcal{U}_2 = \mathcal{U}, \\ \mathcal{X}_1 &= \mathcal{X}_2 = \mathcal{X}. \end{aligned}$$

To simplify notation, we assume without loss of generality that

$$\begin{aligned} \mathcal{U} &= \{0, 1, \dots, U-1\}, & \mathcal{X} &= \{0, 1, \dots, X-1\}, \\ \mathcal{W} &= \{0, 1, \dots, W-1\}, & \mathcal{Y} &= \{0, 1, \dots, Y-1\}. \end{aligned}$$

Finally, to avoid trivial cases, we assume that all cardinalities are strictly bigger than one, and that  $W \leq U^2$ .

We denote by  $\mathcal{A}(U, W)$  the collection of all target functions  $a: \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{W}$ . Similarly, we denote by  $\mathcal{G}(X, Y)$  the collection of all channels  $g: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ . The next example introduces several target functions  $\mathbf{A}$  and channels  $\mathbf{G}$  that will be used to illustrate results in the remainder of the paper.

**Example 1.** We start by introducing four target functions  $a(\cdot, \cdot)$ .

- Let  $\mathcal{W} = \mathcal{U} \times \mathcal{U}$ . The *identity* target function is

$$a(u_1, u_2) \triangleq (u_1, u_2)$$

for all  $u_1, u_2 \in \mathcal{U}$ . Since we will refer to the identity target function repeatedly, we will denote it by the symbol  $\mathbf{A}_I$ .

- Let  $\mathcal{W} = \{0, 1\}$ . The *equality* target function is

$$a(u_1, u_2) \triangleq \begin{cases} 1, & \text{if } u_1 = u_2 \\ 0, & \text{otherwise} \end{cases}$$

for all  $u_1, u_2 \in \mathcal{U}$ .

- Let  $\mathcal{W} = \{0, 1\}$ . The *greater-than* target function is

$$a(u_1, u_2) \triangleq \begin{cases} 1, & \text{if } u_1 > u_2 \\ 0, & \text{otherwise} \end{cases}$$

for all  $u_1, u_2 \in \mathcal{U}$ .

- A *random* target function corresponds to the matrix  $\mathbf{A}$  being a random variable, with each entry chosen independently and uniformly over  $\mathcal{W}$ . The matrix  $\mathbf{A}$  is generated before communication begins and is known at both the transmitters and at the receiver.

We now introduce three channels  $g(\cdot, \cdot)$ .

- Let  $\mathcal{X} = \{0, 1\}$  and  $\mathcal{Y} = \{0, 1, 2\}$ . The *binary adder* MAC is given by

$$g(x_1, x_2) \triangleq x_1 + x_2$$

for all  $x_1, x_2 \in \mathcal{X}$ , and where  $+$  denotes ordinary addition.

- Let  $\mathcal{X} = \{0, 1\}$  and  $\mathcal{Y} = \{0, 1\}$ . The *Boolean*  $\vee$  or *Boolean OR* MAC is

$$g(x_1, x_2) \triangleq \begin{cases} 0, & \text{if } x_1 = x_2 = 0 \\ 1, & \text{otherwise} \end{cases}$$

for all  $x_1, x_2 \in \mathcal{X}$ .

- A *random* channel corresponds to the matrix  $\mathbf{G}$  being a random variable, with each entry chosen independently and uniformly over  $\mathcal{Y}$ . The matrix  $\mathbf{G}$  is generated before communication begins and is known at both the transmitters and at the receiver.

◇

The emphasis in this paper is on the asymptotic behavior for large function domains, i.e., as  $U \rightarrow \infty$ . We allow the other cardinalities  $X(U)$ ,  $Y(U)$  and  $W(U)$  to scale as a function of  $U$ . We use the notation

$$X(U) \dot{\leq} U^a$$

for the relation

$$\limsup_{U \rightarrow \infty} \frac{\log(X(U))}{\log(U)} \leq a$$

and analogously for  $\dot{<}$ . Similarly, we use

$$X(U) \dot{\geq} U^a$$

for the relation

$$\liminf_{U \rightarrow \infty} \frac{\log(X(U))}{\log(U)} \geq a$$

and analogously for  $\dot{>}$ . Finally,

$$X(U) \dot{=} U^a$$

is short hand for

$$X(U) \dot{\leq} U^a \quad \text{and} \quad X(U) \dot{\geq} U^a.$$

For example,  $X(U) \dot{=} U^a$  is equivalent<sup>3</sup> to  $X(U) = U^{a \pm o(1)}$  as  $U \rightarrow \infty$ . With slight abuse of notation, we will write  $X(U) \dot{\leq} U^\infty$  to mean that  $X(U) \dot{\leq} U^\eta$  for *some* finite  $\eta$ .

Throughout this paper, we are interested in efficient computation of the target function  $a(\cdot, \cdot)$  over the channel  $g(\cdot, \cdot)$ . In Theorems 1 and 2 only a single use of the channel is permitted, and efficiency is expressed in terms of the required cardinalities  $X(U)$  and  $Y(U)$  of the channel alphabets as a function of  $U$ . In Theorems 3 and 4, multiple uses of the channel are allowed, and efficiency is then naturally expressed in terms of the number of required channel uses  $n(U)$  as a function of  $U$ .

Finally, all results are stated in terms of the fraction of channels (in Theorems 1 and 2) or target functions (in Theorem 4) for which successful computation is possible. The proofs of all the theorems are based on probabilistic methods by using a uniform distribution over choices of channel  $g(\cdot, \cdot)$  or target functions  $a(\cdot, \cdot)$ .

### III. MAIN RESULTS

Let  $\mathbf{A}_I \in \mathcal{A}(U, U^2)$  be the identity target function introduced in Example 1, and let  $\mathbf{G}$  be an arbitrary channel matrix. Consider any other target function  $\mathbf{A} \in \mathcal{A}(U, W)$  over the same domain  $\mathcal{U} \times \mathcal{U}$ , but with possibly different range  $\mathcal{W}$ . Assume  $(\mathbf{A}_I, \mathbf{G})$  is  $\delta$ -feasible. Then  $(\mathbf{A}, \mathbf{G})$  is also  $\delta$ -feasible, since we can first compute  $\mathbf{A}_I$  (and hence  $\hat{u}_1$  and  $\hat{u}_2$ ) over the channel  $\mathbf{G}$  and then simply apply the function  $\mathbf{A}$  to the recovered messages  $\hat{u}_1$  and  $\hat{u}_2$ . This architecture, separating the computation task from the communication task, is illustrated in Fig. 2.

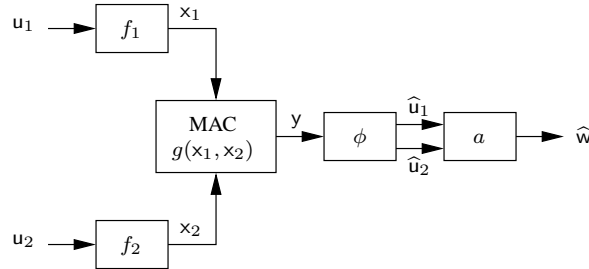


Fig. 2. Separation-based scheme computing the function  $a(\cdot, \cdot)$  over the MAC  $g(\cdot, \cdot)$ . The receiver first decodes the original messages  $(\hat{u}_1, \hat{u}_2)$  and then evaluates the desired target function  $a(\hat{u}_1, \hat{u}_2)$ .

As a concrete example, let  $\mathbf{A}$  be the greater-than target function introduced in Example 1. The range  $\mathcal{W} = \{0, 1\}$  of  $\mathbf{A}$  has cardinality two. On the other hand, the identity function  $\mathbf{A}_I$  has range  $\mathcal{U} \times \mathcal{U}$  of cardinality  $U^2$ . In other words, for large  $U$ , the identity target function is considerably more complicated than the greater-than target function. As a result, one might expect that the separation-based architecture in Fig. 2 is highly suboptimal in terms of the computation efficiency as described in Section II. As the main result of this paper, we prove that this intuition is wrong in most cases. Instead, we show that for most pairs  $(\mathbf{A}, \mathbf{G})$  of target function and MAC, separation of computation and communication is close to optimal.

We discuss the single channel-use case in Section III-A, and the  $n$  channel-uses case in Section III-B.

#### A. Single Channel Use ( $n = 1$ )

In this section, we will focus on the case where the target function needs to be computed using just one use of the channel. The natural value of the upper bound on the probability of error is  $\delta = 0$  in this case. In other words, we will be interested in 0-feasibility.

<sup>3</sup>Note that the notation  $f(U)$  is  $o(1)$  as  $U \rightarrow \infty$  stands for  $\lim_{U \rightarrow \infty} f(U) = 0$ .

We start by deriving conditions under which computation of the identity target function over a MAC is feasible. Equivalently, these conditions guarantee that *any* target function with same domain cardinality  $U$  can be computed over a MAC by separating communication and computation as discussed above.

**Theorem 1.** *Let  $\mathbf{A}_I \in \mathcal{A}(U, U^2)$  be the identity target function, and assume*

$$X(U) \succ U, \quad (1a)$$

$$Y(U) \succ U^3. \quad (1b)$$

Then,

$$\lim_{U \rightarrow \infty} \frac{|\{\mathbf{G} \in \mathcal{G}(X(U), Y(U)) : (\mathbf{A}_I, \mathbf{G}) \text{ is 0-feasible}\}|}{|\mathcal{G}(X(U), Y(U))|} = 1.$$

The proof of Theorem 1 is reported in Section IV-B. Recall that  $\mathcal{G}(X, Y)$  is the collection of all channels  $\mathbf{G}$  of dimension  $X \times X$  and range of cardinality  $Y$ . Theorem 1 (together with the separation approach discussed earlier) thus roughly implies that any target function with a domain of cardinality  $U$  can be computed over most MACs of input cardinality  $X(U)$  of order at least  $U$  and output cardinality  $Y(U)$  of order at least  $U^3$ . The precise meaning of “most” is that the fraction of channels  $\mathbf{G}$  in  $\mathcal{G}(X, Y)$  for which the statement holds goes to one as  $U \rightarrow \infty$ . A look at the proof of the theorem shows that the convergence to this limit is, in fact, exponentially fast. In other words, the fraction of channels for which the theorem fails to hold is exponentially small in the domain cardinality  $U$ .

Since the achievable scheme is separation based, this conclusion holds regardless of the cardinality  $W(U)$  of the range of the target function. Similarly, since it is clear that the channel input has to have at least cardinality  $X(U)$  of order  $U$  for successful computation, we see that the condition on  $X(U)$  in Theorem 1 is not a significant restriction. What is significant, however, is the restriction that  $Y(U)$  is at least of order  $U^3$ . The next result shows that this restriction on  $Y(U)$  is essentially also necessary.

Before we state the theorem, we need to introduce one more concept.

**Definition.** Consider a target function  $a: \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{W}$ . For a set  $\widetilde{\mathcal{W}} \subset \mathcal{W}$ , consider

$$a^{-1}(\widetilde{\mathcal{W}}) \triangleq \{(u_1, u_2) \in \mathcal{U} \times \mathcal{U} : a(u_1, u_2) \in \widetilde{\mathcal{W}}\}.$$

For  $c \in (0, 1/2]$ , the target function  $a(\cdot, \cdot)$  is said to be  $c$ -balanced if there exist a partition  $\mathcal{W}_1, \mathcal{W}_2$  of  $\mathcal{W}$  such that

$$|a^{-1}(\mathcal{W}_i)| \geq c \cdot U^2$$

for all  $i \in \{1, 2\}$ .

Most functions are  $c$ -balanced for any  $c < 1/3$  and  $W(U)$  as long as  $U$  is large enough. Indeed, choosing  $\mathcal{W}_1 = \{0, \dots, \lfloor W(U)/2 \rfloor - 1\}$  and  $\mathcal{W}_2 = \{\lfloor W(U)/2 \rfloor, \dots, W(U) - 1\}$  shows that

$$\lim_{U \rightarrow \infty} \frac{|\{\mathbf{A} \in \mathcal{A}(U, W(U)) : \mathbf{A} \text{ is } 1/3\text{-balanced}\}|}{|\mathcal{A}(U, W(U))|} = 1, \quad (2)$$

where we recall that  $\mathcal{A}(U, W)$  denotes the collection of all target functions  $\mathbf{A}$  of dimension  $U \times U$  and range of cardinality  $W$ . In fact, the convergence in (2) is again exponentially fast<sup>4</sup> in  $U$ . Moreover, many functions of specific interest are balanced.

**Example 2.** Consider the target functions introduced in Example 1.

- The identity and the greater-than target functions are  $c$ -balanced for any constant  $c < 1/2$  and  $U$  large enough.
- The equality target function is *not*  $c$ -balanced for any constant  $c > 0$  as  $U \rightarrow \infty$ . Indeed, since  $W(U) = 2$  in this case, the only choice (up to labeling) is to set  $\mathcal{W}_1 = \{0\}$  and  $\mathcal{W}_2 = \{1\}$ . Then  $|a^{-1}(\mathcal{W}_1)| = U^2 - U$  and  $|a^{-1}(\mathcal{W}_2)| = U$ , which is not  $c$ -balanced for any constant  $c > 0$  as  $U \rightarrow \infty$ .

<sup>4</sup>This follows directly from results on the convergence of empirical distributions.

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We have the following converse result to Theorem 1 for balanced target functions.

**Theorem 2.** Fix a constant  $c \in (0, 1/2]$  independent of  $U$ . Assume  $W(U) \geq 2$  and

$$X(U) \prec U^\infty, \quad (3a)$$

$$Y(U) \prec U^3. \quad (3b)$$

Let  $\mathbf{A} \in \mathcal{A}(U, W(U))$  be any  $c$ -balanced target function. Then

$$\lim_{U \rightarrow \infty} \frac{|\{\mathbf{G} \in \mathcal{G}(X(U), Y(U)) : (\mathbf{A}, \mathbf{G}) \text{ is 0-feasible}\}|}{|\mathcal{G}(X(U), Y(U))|} = 0.$$

The proof of Theorem 2 is reported in Section IV-C. Recall that the notation  $X(U) \prec U^\infty$  is used to indicate that  $X(U)$  grows at most polynomially in  $U$ —an assumption that is quite mild. Thus, Theorem 2 roughly states that regardless of the value of  $W(U)$ , if the cardinality  $Y(U)$  of the channel output is order-wise less than  $U^3$ , then any balanced target function with a range of cardinality  $W(U)$  cannot be computed over most MACs. Here the precise meaning of “most” is again that the fraction of channel matrices with at most  $Y(U)$  channel outputs for which successful computation is possible converges to zero, and a look at the proof reveals again that this convergence is, in fact, exponentially fast in  $U$ .

Comparing this to Theorem 1, we see that the same scaling of  $Y(U)$  allows computation of a target function using a separation based scheme (i.e., by first recovering the two messages  $(\hat{u}_1, \hat{u}_2)$  and then applying the target function to compute the estimate  $\hat{w} = a(\hat{u}_1, \hat{u}_2)$ ). Thus, for the computation of a given balanced function over most MACs, separation of computation and communication is essentially optimal. Moreover, since most functions are balanced by (2), the same also holds for most pairs  $(\mathbf{A}, \mathbf{G})$  of target and channel functions.

**Example 3.** Let  $\mathbf{A}$  be the greater-than target function of domain  $U \times U$  introduced in Example 1. Note that this target function has range of cardinality  $W(U) = 2$ , i.e.,  $\mathbf{A}$  is binary. From Example 2, we know that  $\mathbf{A}$  is balanced for any constant  $c < 1/2$  and  $U$  large enough. Thus Theorem 2 applies, showing that, for large  $U$  and most MACs  $\mathbf{G}$ , separation of computation and communication is essentially optimal.

Observe that the receiver is interested in only a *single* bit of information about  $(u_1, u_2)$ . Nevertheless, the structure of the greater-than target function is complicated enough that, in order to recover this single bit, the decoder is essentially forced to learn  $(u_1, u_2)$  itself. In other words, in order to compute the single desired bit, communication of  $2 \log(U)$  message bits is essentially necessary. ◇

Theorem 2 is restricted to balanced functions. Even though only a vanishingly small fraction of target functions is not balanced, it is important to understand this restriction. We illustrate this through the following example.

**Example 4.** Assume  $W(U) = 2$  and

$$X(U) \succ U, \quad (4a)$$

$$Y(U) \succ U. \quad (4b)$$

Let  $\mathbf{A}_= \in \mathcal{A}(U, 2)$  be the equality target function introduced in Example 1. Then

$$\lim_{U \rightarrow \infty} \frac{|\{\mathbf{G} \in \mathcal{G}(X(U), Y(U)) : (\mathbf{A}_=, \mathbf{G}) \text{ is 0-feasible}\}|}{|\mathcal{G}(X(U), Y(U))|} = 1. \quad (5)$$

The proof of the above statement is reported in Section IV-D. This result shows that the equality function can be computed over a large fraction of MACs with output cardinality  $Y(U)$  of order at least  $U$ . This contrasts with output cardinality  $Y(U)$  of order  $U^3$  that is required for successful computation of balanced functions in Theorem 2. Recall from Example 2 that the equality target function is *not*  $c$ -balanced for any  $c > 0$  and  $U$  large enough. Thus, (5) does not contradict Theorem 2. It does, however, show that for unbalanced functions separation of communication and computation can be suboptimal. ◇



### B. Multiple Channel Uses ( $n \geq 1$ )

In this section, we allow multiple uses of the MAC. Our emphasis will again be on the asymptotic behavior for large function domains  $U \rightarrow \infty$ . However, in this section we keep the MAC  $g(\cdot, \cdot)$ , and hence also the cardinalities of the channel domain  $\mathcal{X}$  and channel range  $\mathcal{Y}$ , fixed. Instead, we characterize the minimum number  $n = n(U)$  of channel uses required to compute the target function.

We begin by stating a result for the identity target function introduced in Example 1. Equivalently, this result applies to *any* target function (with same domain cardinality  $U$ ) by using a scheme separating communication and computation. Let  $H(x)$  denote the entropy of a random variable  $x$ .

**Theorem 3.** Fix a constant  $\delta > 0$  independent of  $U$ , and assume that  $X$  and  $Y$  are constant. Let  $\mathbf{A}_I \in \mathcal{A}(U, U^2)$  be the identity target function, and let  $\mathbf{G} \in \mathcal{G}(X, Y)$  be any MAC. Consider any joint distribution of the form  $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$ , where  $p(y|x_1, x_2)$  is specified by the channel function  $\mathbf{G}$ . Then, for any  $n(U)$  satisfying

$$U^2 \leq 2^{n(U)H(y|q)}, \quad (6a)$$

$$U \leq 2^{n(U)H(y|x_1, q)}, \quad (6b)$$

$$U \leq 2^{n(U)H(y|x_2, q)}, \quad (6c)$$

$(\mathbf{A}_I, \mathbf{G}^{\otimes n(U)})$  is  $\delta$ -feasible for  $U$  large enough.

The result follows directly from the characterization of the achievable rate region for ordinary communication over the MAC, see for example [18, Theorem 14.3.3]. Using separation, Theorem 3 implies that, for large enough  $U$ , any target function of domain cardinality  $U$  can be reliably computed over  $n(U)$  uses of an MAC  $\mathbf{G}$  as long as it satisfies the constraints in (6). The next result states that for most functions these restrictions on  $n(U)$  are essentially also necessary.

**Theorem 4.** Assume that<sup>5</sup>

$$W(U) \geq \omega(1)$$

as  $U \rightarrow \infty$ , that  $0 < \delta \leq 1/(2 \ln(W(U)))$ , and that  $X$  and  $Y$  are constant. Let  $\mathbf{G} \in \mathcal{G}(X, Y)$  be any MAC. Then, for any  $n(U)$  satisfying

$$\lim_{U \rightarrow \infty} \frac{|\{\mathbf{A} \in \mathcal{A}(U, W(U)) : (\mathbf{A}, \mathbf{G}^{\otimes n(U)}) \text{ is } \delta\text{-feasible}\}|}{|\mathcal{A}(U, W(U))|} > 0,$$

we must have

$$U^2 \leq 2^{n(U)H(y|q)},$$

$$U \leq 2^{n(U)H(y|x_1, q)},$$

$$U \leq 2^{n(U)H(y|x_2, q)}$$

for some joint distribution of the form  $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$ , where  $p(y|x_1, x_2)$  is specified by the channel function  $\mathbf{G}$ .

The proof of Theorem 4 is presented in Section IV-E. Recall that  $\mathcal{A}(U, W)$  denotes the collection of all target functions  $\mathbf{A}$  of dimension  $U \times U$  and range of cardinality  $W$ . Together, Theorems 3 and 4 thus show that, for any deterministic MAC and most target functions, the smallest number of channel uses  $n^*(U)$  that enables reliable computation is of the same order as that needed for the identity function. Moreover, they show that for most such pairs, separation of communication and computation is essentially optimal even if we allow multiple uses of the channel and nonzero error probability. Here the precise meaning of “most” is that the statement holds for all but a vanishing fraction of functions. Moreover, the proof of the theorem shows again that this fraction is, in fact, exponentially small in  $U$ .

<sup>5</sup>Note that the notation  $W(U) \geq \omega(1)$  as  $U \rightarrow \infty$  stands for  $\lim_{U \rightarrow \infty} W(U) = \infty$ .

**Example 5.** Let  $\mathbf{G}$  be the binary adder MAC introduced in Example 1. Define

$$H^*(\mathbf{G}) \triangleq \max_{\mathbf{x}_1, \mathbf{x}_2} H(g(\mathbf{x}_1, \mathbf{x}_2)) = 3/2,$$

where the maximization is over all independent random variables  $\mathbf{x}_1, \mathbf{x}_2$  taking values in the channel input alphabet  $\mathcal{X}$ .  $H^*(\mathbf{G})$  denotes the maximum entropy that can be induced at the channel output. For the binary adder MAC  $\mathbf{G}$ , it follows from (6) in Theorem 3 that the identity function can be reliably computed over  $\mathbf{G}$  if the number of channel uses  $n(U) \geq 2 \log U / H^*(\mathbf{G}) = 4 \log U / 3$ . On the other hand, Theorem 4 shows that for most functions  $\mathbf{A}$  of domain  $U \times U$  and range cardinality  $W(U) = \log(U)$ , the smallest number of channel uses  $n^*(U)$  required for reliable computation is of order  $4 \log(U) / 3$ . Thus, near-optimal performance can be achieved by separating computation and communication. In other words, even though the receiver is only interested in  $\log \log(U)$  function bits, it is essentially forced to learn the  $2 \log(U)$  message bits as well.

This example also illustrates that the usual way of proving converse results based on the cut-set bound is not tight for most  $(\mathbf{A}, \mathbf{G})$  pairs. For example, [14, Lemma 13] shows that for reliable computation we need to have

$$n(U)H^*(\mathbf{G}) \geq H(a(\mathbf{u}_1, \mathbf{u}_2))$$

where  $H(\cdot)$  denotes entropy. Since  $\mathbf{A}$  has range of cardinality  $W(U)$ , we have

$$H(a(\mathbf{u}_1, \mathbf{u}_2)) \leq \log(W(U)).$$

For  $W(U) = \log(U)$  and  $H^*(\mathbf{G}) = 3/2$  as considered here, the tightest bound that can in the *best case* be derived via the cut-set approach is thus

$$n^*(U) \geq \log(W(U)) / H^*(\mathbf{G}) = 2 \log \log(U) / 3.$$

However, we know that the correct scaling for  $n^*(U)$  is  $4 \log(U) / 3$ . Hence, the cut-set bound is loose by an unbounded factor as  $U \rightarrow \infty$ .  $\diamond$

## IV. PROOFS

We now prove the main results. The proofs of Theorems 1 and 2 are reported in Sections IV-B and IV-C respectively. The proof of (5) in Example 4 is presented in Section IV-D. Finally, the proof of Theorem 3 is covered in Section IV-E. We start by presenting some preliminary observations in Section IV-A.

### A. Preliminaries

Recall our assumption that no two rows or two columns of the target function  $\mathbf{A}$  are identical. As a result,  $\mathbf{A}$  can be computed over the channel  $\mathbf{G}$  with zero error, i.e.,  $(\mathbf{A}, \mathbf{G})$  is 0-feasible, if and only if there exists a  $U \times U$  submatrix (with ordered rows and columns)  $\mathbf{S}$  of  $\mathbf{G}$  such that any two entries  $(u_1, u_2)$  and  $(\tilde{u}_1, \tilde{u}_2)$  with  $a_{u_1, u_2} \neq a_{\tilde{u}_1, \tilde{u}_2}$  satisfy  $s_{u_1, u_2} \neq s_{\tilde{u}_1, \tilde{u}_2}$ , see Fig. 3.

On the other hand, this is not necessary if a probability of error  $\delta > 0$  can be tolerated. As an example, consider the equality function. For any positive  $\delta$ , a trivial decoder that always outputs 0 computes the equality function with probability of error  $1/U$ . As  $U \rightarrow \infty$ , the probability of error is eventually less than  $\delta$ . This motivates the following definition.

Given a target function  $a: \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{W}$ , a function  $a_\delta: \mathcal{V}_1 \times \mathcal{V}_2 \rightarrow \mathcal{W}$  with

$$V_1 \triangleq |\mathcal{V}_1| \leq U,$$

$$V_2 \triangleq |\mathcal{V}_2| \leq U$$

is said to be a  $\delta$ -approximation of  $a(\cdot, \cdot)$  if there exist two mappings  $f_1: \mathcal{U} \rightarrow \mathcal{V}_1$  and  $f_2: \mathcal{U} \rightarrow \mathcal{V}_2$  such that

$$\left| \left\{ (u_1, u_2) \in \mathcal{U} \times \mathcal{U} : a(u_1, u_2) \neq a_\delta(f_1(u_1), f_2(u_2)) \right\} \right| \leq \delta U^2. \quad (7)$$

	0	1		0	1		00	01	10	11
0	1	0	0	0	1	00	00	01	10	11
1	0	1	1	1	1	01	01	<b>01</b>	<b>11</b>	11
						10	10	<b>11</b>	<b>10</b>	11
						11	11	11	11	11

$\mathbf{A}$                        $\mathbf{G}$                        $\mathbf{G}^{\otimes 2}$

Fig. 3. Structure of a zero-error computation scheme over a MAC. The target function  $\mathbf{A}$  corresponds to the equality function, the MAC matrix  $\mathbf{G}$  corresponds to the Boolean  $\vee$  MAC, and  $\mathbf{G}^{\otimes 2}$  denotes the 2-fold use of channel  $\mathbf{G}$ . While the function  $\mathbf{A}$  cannot be computed over  $\mathbf{G}$  in one channel use, it can be computed in two channel uses by assigning the channel input 01 to user message 0 and 10 to user message 1. The corresponding ordered submatrix  $\mathbf{S}$  of  $\mathbf{G}$  is indicated in bold lines in the figure.

In words, the target function  $a(\cdot, \cdot)$  is equal to the approximation function  $a_\delta(\cdot, \cdot)$  for at least a  $(1 - \delta)$  fraction of all message pairs. As before, a  $\delta$ -approximation function  $a_\delta$  can be represented by a  $V_1 \times V_2$  matrix  $\mathbf{A}_\delta$ . We have the following straightforward relation.

**Lemma 5.** *Consider a target function  $\mathbf{A}$  and MAC  $\mathbf{G}$ . If  $(\mathbf{A}, \mathbf{G})$  is  $\delta$ -feasible, then there exists a  $\delta$ -approximation  $\mathbf{A}_\delta$  of  $\mathbf{A}$  such that  $(\mathbf{A}_\delta, \mathbf{G})$  is 0-feasible.*

*Proof:* Let  $f_1, f_2$  and  $\phi$  be the encoders and the decoder achieving probability of error at most  $\delta$  for  $(\mathbf{A}, \mathbf{G})$ . Let  $\mathcal{V}_i$  be the range of  $f_i$ , and set

$$a_\delta(f_1(u_1), f_2(u_2)) \triangleq \phi(g(f_1(u_1), f_2(u_2)))$$

for all  $u_1, u_2 \in \mathcal{U}$ . Then  $a_\delta(\cdot, \cdot)$  is a  $\delta$ -approximation of  $a(\cdot, \cdot)$ , and  $(a_\delta, g)$  is 0-feasible. ■

We will make frequent use of the Chernoff bound, which we recall here for future reference. Let  $z_1, z_2, \dots, z_N$  be independent random variables, and let

$$z \triangleq \sum_{i=1}^N z_i.$$

By Markov's inequality,

$$\mathbb{P}(z > b) < \min_{t>0} \exp(-tb) \prod_{i=1}^N \mathbb{E}(\exp(tz_i)) \quad (8)$$

Assume furthermore that each  $z_i$  takes value in  $\{0, 1\}$ , and set

$$\mu \triangleq \mathbb{E}(z).$$

Then, for any  $\gamma > 0$ ,

$$\mathbb{P}(z > (1 + \gamma)\mu) < \left( \frac{e^\gamma}{(1 + \gamma)^{(1+\gamma)}} \right)^\mu, \quad (9)$$

and, for  $0 < \gamma \leq 1$ ,

$$\mathbb{P}(z < (1 - \gamma)\mu) < \exp(-\mu\gamma^2/2), \quad (10)$$

see for example [19, Theorem 4.1, Theorem 4.2].



there are at least  $Z_U(X)$  edges in  $B$ . By (11),

$$\begin{aligned} Z_U(X) &< (U-1)^{1/U}(X-U+1)X^{1-1/U} + (U-1)X + 1 \\ &= X(X-1) + 1 + X(X-U+1)\left(\left(\frac{U-1}{X}\right)^{1/U} - \frac{X-U}{X-U+1}\right). \end{aligned} \quad (12)$$

Using the inequality  $(1-x)^n \geq 1-nx$  for  $x \in [0, 1]$  and that  $X = 4U$  by assumption, we have

$$\begin{aligned} \left(\frac{X-U}{X-U+1}\right)^U &= \left(1 - \frac{1}{X-U+1}\right)^U \\ &\geq 1 - \frac{U}{X-U+1} = \frac{2U+1}{X-U+1} \\ &\geq \frac{U-1}{X}. \end{aligned}$$

Combining this with (12) shows that

$$Z_U(X) < X^2 - X + 1.$$

Thus we have shown that the identity target function can be computed over *any* channel  $\mathbf{G}$  with  $X = 4U$  if it has at least  $X^2 - X + 1$  distinct entries. Consider now a *random* channel  $\mathbf{G}$ . The next lemma shows that  $\mathbf{G}$  satisfies this condition with high probability as  $X \rightarrow \infty$ .

**Lemma 6.** *Let  $N$  be the number of distinct entries in the random channel matrix  $\mathbf{G}$ , and assume  $Y \geq e^3 X^3$ . Then*

$$\mathbb{P}(N \geq X^2 - X + 1) \geq 1 - \exp(-(X-2))$$

for  $X$  large enough.

The proof of Lemma 6 is reported in Appendix A. Lemma 6 shows that with probability at least  $1 - \exp(-(X-2))$  the identity target function can be computed with zero error over the random MAC  $\mathbf{G}$ . Since  $X = 4U$  so that  $X \rightarrow \infty$  as  $U \rightarrow \infty$ , the statement of the theorem follows. ■

### C. Proof of Theorem 2

Let  $\mathcal{W}_1$  and  $\mathcal{W}_2$  be the two sets in the definition of a balanced function. For a MAC  $\mathbf{G}$ , a  $U \times U$  ordered submatrix  $\mathbf{S}$  corresponds to a valid code for computing  $\mathbf{A}$  with zero error only if there are no common values between the entries in  $\mathbf{S}$  corresponding to  $a^{-1}(\mathcal{W}_1)$  and  $a^{-1}(\mathcal{W}_2)$ . Consider then a random  $\mathbf{G}$  (as introduced in Example 1) and one such ordered submatrix  $\mathbf{S}$ . Observe that the selection of rows and columns of  $\mathbf{G}$  in  $\mathbf{S}$  is fixed—the matrix  $\mathbf{S}$  is random because its entries are derived from the random matrix  $\mathbf{G}$ . Let  $N_1$  denote the number of distinct values among the entries corresponding to  $a^{-1}(\mathcal{W}_1)$  in  $\mathbf{S}$ . We have the following bound on  $N_1$ .

**Lemma 7.** *Assume  $a^{-1}(\mathcal{W}_1) \geq cU^2$ , and set*

$$N \triangleq \min\{Y/3, cU^2/3\}.$$

Then

$$\mathbb{P}(N_1 < N) \leq \exp(-cU^2/6).$$

The proof of Lemma 7 is reported in Appendix B. The submatrix  $\mathbf{S}$  corresponds to a valid code for computing the target function  $\mathbf{A}$  only if all the entries corresponding to  $a^{-1}(\mathcal{W}_2)$  take values from the

$Y - N_1$  channel outputs not present in the entries corresponding to  $a^{-1}(\mathcal{W}_1)$ . Thus the probability of the submatrix  $\mathbf{S}$  being a valid code for computing the target function  $\mathbf{A}$  is at most

$$\begin{aligned} \mathbb{P}(\mathbf{S} \text{ is a valid code for } \mathbf{A}) &\leq \mathbb{P}(N_1 < N) + \left(\frac{Y - N}{Y}\right)^{|a^{-1}(\mathcal{W}_2)|} \\ &\stackrel{(a)}{\leq} \exp(-cU^2/6) + \exp(-N|a^{-1}(\mathcal{W}_2)|/Y) \\ &\stackrel{(b)}{\leq} \exp(-cU^2/6) + \exp(-NcU^2/Y) \\ &\stackrel{(c)}{\leq} 2 \exp(-\min\{cU^2/6, c^2U^4/(3Y)\}), \end{aligned} \quad (13)$$

where (a) follows from Lemma 7 and  $1 - x \leq e^{-x}$ , (b) follows since  $\mathbf{A}$  is  $c$ -balanced, and (c) follows from the definition of  $N$ .

The pair  $(\mathbf{A}, \mathbf{G})$  is 0-feasible if and only if there exists some valid ordered submatrix  $\mathbf{S}$  with dimension  $U \times U$  of the  $X \times X$  channel matrix  $\mathbf{G}$ . There are at most  $X^{2U}$  ways to choose the rows and columns of this submatrix. Hence, from the union bound and (13), the probability that  $(\mathbf{A}, \mathbf{G})$  is 0-feasible is at most

$$\begin{aligned} \mathbb{P}((\mathbf{A}, \mathbf{G}) \text{ is 0-feasible}) &\leq X^{2U} \mathbb{P}(\mathbf{S} \text{ is a valid code for } \mathbf{A}) \\ &\leq X^{2U} \cdot 2 \exp(-\min\{cU^2/6, c^2U^4/(3Y)\}) \\ &= \exp(-(\min\{cU^2/6, c^2U^4/(3Y)\} - 2U \ln(X) - \ln(2))). \end{aligned}$$

Now, (3) implies that  $X \leq U^m$  and  $Y \leq c^2U^3/(12m \ln(U))$  for some finite positive  $m$  and  $U$  large enough. Hence,

$$\lim_{U \rightarrow \infty} \mathbb{P}((\mathbf{A}, \mathbf{G}) \text{ is 0-feasible}) = 0,$$

as needed to be shown. ■

#### D. Proof of (5) in Example 4

A scheme computes the equality target function  $\mathbf{A}_=$  with zero error if and only if the channel outputs corresponding to the set of message pairs  $\{(u, u) : u \in \mathcal{U}\}$  are all distinct from those corresponding to the message pairs  $\{(u_1, u_2) : u_1 \neq u_2\}$ . In what follows, we will first show that this is guaranteed if the channel matrix  $\mathbf{G}$  satisfies certain properties. We then argue that a random channel matrix  $\mathbf{G}$  (as introduced in Example 1) satisfies these properties with high probability.

From (4), we can assume that  $X \geq 200U \ln(U)$  and  $Y \geq 16U$  for  $U$  large enough. We will prove the result under these two weaker conditions. Since the encoders can always choose to ignore some of the channel inputs, we can assume without loss of generality that  $X = 200U \ln(U)$ . Throughout this proof, we denote by  $k$  the largest integer such that  $Y/k \geq 16U$ . Note that this implies

$$16U \leq Y/k < 32U. \quad (14)$$

Given an arbitrary MAC  $\mathbf{G}$ , create a bipartite graph  $B$  as follows. Let the vertices on the two sides of the bipartite graph correspond to the  $X$  different row and column indices of the channel matrix  $\mathbf{G}$ . Fix an arbitrary subset  $\tilde{\mathcal{Y}}$  of cardinality  $k$  of  $\mathcal{Y}$ . Place an edge in the bipartite graph  $B$  between a node  $x_1$  on the left and a node  $x_2$  on the right if  $g(x_1, x_2) \in \tilde{\mathcal{Y}}$ , see Fig. 5.

An *induced matching*  $M$  in a bipartite graph  $B$  is a subset of edges such that i) no pair of edges in  $M$  share a common endpoint and ii) no pair of edges in  $M$  are joined by an edge in  $B$ . Note that an induced matching  $M$  of size  $U$  in  $B$  corresponds to a zero-error computation scheme for the equality function  $\mathbf{A}_=$ . This follows from the observation that the induced matching provides a subset of channel inputs  $\{x_{1,1}, x_{1,2}, \dots, x_{1,U}\} \subset \mathcal{X}_1$  and  $\{x_{2,1}, x_{2,2}, \dots, x_{2,U}\} \subset \mathcal{X}_2$  such that the only pairs of channel inputs for which the channel output is in  $\tilde{\mathcal{Y}}$  are given by  $\{(x_{1,k}, x_{2,k}) : k \in \{1, 2, \dots, U\}\}$ . The decoder thus simply maps all channel outputs in  $\tilde{\mathcal{Y}}$  to 1 and all other channel outputs to 0.

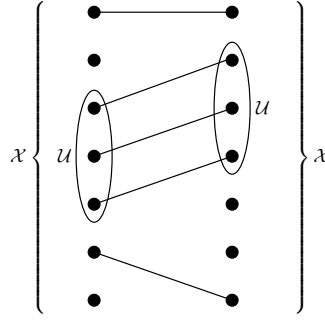


Fig. 5. Bipartite graph  $B$  representing the channel matrix  $\mathbf{G}$ . The left vertices correspond to the possible channel inputs at transmitter one, and the right vertices correspond to the possible channel inputs at transmitter two. For some fixed subset  $\tilde{\mathcal{Y}} \subseteq \mathcal{Y}$  of cardinality  $k$ , the graph contains an edge between two vertices  $x_1, x_2$  if the corresponding channel output  $g(x_1, x_2)$  is an element of  $\tilde{\mathcal{Y}}$ . The existence of an induced matching of size  $U$  corresponds to a scheme for computing the equality target function.

A *strong edge-coloring* of a graph  $B$  is an edge-coloring in which every color class is an induced matching, i.e., any two vertices belonging to distinct edges with the same color are not adjacent. The strong chromatic index  $\chi_s(B)$  is the minimum number of colors in a strong edge-coloring of  $B$ . A simple argument in [21] shows that for any graph  $B$ ,

$$\chi_s(B) \leq 2\Delta^2(B),$$

where  $\Delta(B)$  is the maximum degree of  $B$ . Thus, a graph  $B$  contains an induced matching of size at least

$$\frac{m(B)}{\chi_s(B)} \geq \frac{m(B)}{2\Delta^2(B)} \quad (15)$$

where  $m(B)$  denotes the number of edges in  $B$ .

Consider again the bipartite graph  $B$  constructed from  $\mathbf{G}$  for some fixed subset  $\tilde{\mathcal{Y}}$ . From (15) and the above discussion, we see that  $(\mathbf{A}_=, \mathbf{G})$  is 0-feasible if

$$\frac{m(B)}{2\Delta^2(B)} \geq U. \quad (16)$$

We now show that this holds with high probability for a random channel matrix  $\mathbf{G}$ . Since we consider a random  $\mathbf{G}$ , the graph  $B$  is itself also random.

We start by deriving a lower bound on the number of edges  $m(B)$  in  $B$ . The event that a particular pair of vertices has an edge in  $B$  is equivalent to the corresponding entry in the channel matrix  $\mathbf{G}$  being an element of  $\tilde{\mathcal{Y}}$ , which happens with probability  $k/Y$ . Since the  $X^2$  entries of  $\mathbf{G}$  are independent, this implies that the number of edges  $m(B)$  is given by a binomial random variable with mean  $kX^2/Y$ . Thus, using the Chernoff bound (10), that  $X = 200U \ln(U)$  by assumption, and that  $Y/k < 32U$  by (14),

$$\begin{aligned} \mathbb{P}(m(B) < kX^2/(2Y)) &< \exp(-kX^2/(8Y)) \\ &< \exp(-U \ln^2(U)), \end{aligned} \quad (17)$$

which converges to zero as  $U \rightarrow \infty$ .

We continue by deriving an upper bound on the maximum degree  $\Delta(B)$  of  $B$ . Let  $\Delta_L(B)$  and  $\Delta_R(B)$  denote the maximum degree among the left-side and right-side vertices, respectively. Note that  $\Delta_L(B)$  and  $\Delta_R(B)$  are identically distributed, as the maximum value among  $X$  independent binomial random variables with mean  $kX/Y$ . Let  $z$  be one such binomial random variable. Using the Chernoff bound (9),

$$\begin{aligned} \mathbb{P}(\Delta_L(B) \leq 2kX/Y) &= \mathbb{P}(\Delta_R(B) \leq 2kX/Y) \\ &= (\mathbb{P}(z \leq 2kX/Y))^X \\ &\geq (1 - (e/4)^{kX/Y})^X. \end{aligned}$$

By the union bound, and using that  $X = 200U \ln(U)$  by assumption, that  $Y/k < 32U$  by (14), and that  $e/4 < \exp(-1/3)$ , we have

$$\begin{aligned}
\mathbb{P}(\Delta(\mathbf{B}) > 2kX/Y) &= \mathbb{P}(\{\Delta_L(\mathbf{B}) > 2kX/Y\} \cup \{\Delta_R(\mathbf{B}) > 2kX/Y\}) \\
&\leq 2 \left(1 - (1 - (e/4)^{kX/Y})^X\right) \\
&\leq 2X(e/4)^{kX/Y} \\
&\leq \exp(\ln(400U \ln(U)) - 200U \ln(U)/(3 \cdot 32U)) \\
&= \exp(\ln(400 \ln(U)) - 13 \ln(U)/12),
\end{aligned} \tag{18}$$

which converges to zero as  $U \rightarrow \infty$ . Using  $Y/k \geq 16U$  by (14) and the union bound,

$$\begin{aligned}
\mathbb{P}\left(\frac{m(\mathbf{B})}{2\Delta^2(\mathbf{B})} \geq U\right) &\geq \mathbb{P}\left(\frac{m(\mathbf{B})}{2\Delta^2(\mathbf{B})} \geq \frac{Y}{16k}\right) \\
&\geq \mathbb{P}\left(\{m(\mathbf{B}) \geq kX^2/(2Y)\} \cap \{\Delta(\mathbf{B}) \leq 2kX/Y\}\right) \\
&\geq 1 - \left(\mathbb{P}(\{m(\mathbf{B}) < kX^2/(2Y)\}) + \mathbb{P}(\{\Delta(\mathbf{B}) > 2kX/Y\})\right),
\end{aligned}$$

which, by (17) and (18), converges to one as  $U \rightarrow \infty$ . Combined with (16), this shows that

$$\lim_{U \rightarrow \infty} \mathbb{P}((\mathbf{A}_-, \mathbf{G}) \text{ is 0-feasible}) = 1,$$

thus proving the claim. ■

#### E. Proof of Theorem 4

Consider an arbitrary target function  $\mathbf{A}$  and an arbitrary channel function  $\mathbf{G}$ . Recall the definition of a  $\delta$ -approximation function in (7). From Lemma 5,  $(\mathbf{A}, \mathbf{G}^{\otimes n})$  is  $\delta$ -feasible only if there exists some  $\delta$ -approximation  $\mathbf{A}_\delta \in \mathcal{W}^{V_1 \times V_2}$  of  $\mathbf{A}$  such that  $V_1, V_2 \leq U$  and  $(\mathbf{A}_\delta, \mathbf{G}^{\otimes n})$  is 0-feasible. From the construction in Lemma 5, we can assume without loss of generality that  $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{X}^n$ . Furthermore, we can assume without loss of generality that no two rows and no two columns of  $\mathbf{A}_\delta$  are identical. Hence,  $(\mathbf{A}_\delta, \mathbf{G}^{\otimes n})$  is 0-feasible only if there exists a  $V_1 \times V_2$  ordered submatrix  $\mathbf{S}$  of  $\mathbf{G}^{\otimes n}$  computing  $\mathbf{A}_\delta$ , as described in Section IV-A. In the following, denote by  $s: \mathcal{V}_1 \times \mathcal{V}_2 \rightarrow \mathcal{Y}^n$  the mapping corresponding to  $\mathbf{S}$ .

Consider now such a  $V_1 \times V_2$  ordered submatrix  $\mathbf{S}$  of  $\mathbf{G}^{\otimes n}$ . For any  $\mathcal{T} \subseteq \mathcal{V}_1 \times \mathcal{V}_2$ , let  $s(\mathcal{T}) \subseteq \mathcal{Y}^n$  denote the range of  $s(\cdot, \cdot)$ , with the arguments restricted to the subset  $\mathcal{T}$ . Let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be independent random variables uniformly distributed over  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , respectively. Consider the random vector

$$\mathbf{y} = (y[1] \ \cdots \ y[n]) \triangleq s(\mathbf{v}_1, \mathbf{v}_2).$$

Then, for any  $\mathbf{y} \in \mathcal{Y}^n$ , we have

$$\mathbb{P}(\mathbf{y} = \mathbf{y}) = \frac{|\{(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2 : s(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{y}\}|}{V_1 V_2}, \tag{19}$$

and let  $H_S(\mathbf{y})$  denote the corresponding entropy of the random vector  $\mathbf{y}$ . The next result proves the existence of a “typical” set.

**Lemma 8.** *Let  $\mathbf{S}$  be a  $V_1 \times V_2$ -dimensional ordered submatrix of  $\mathbf{G}^{\otimes n}$ , and let  $s: \mathcal{V}_1 \times \mathcal{V}_2 \rightarrow \mathcal{Y}^n$  be the corresponding mapping. For any  $\varepsilon > 0$ , there exists a set  $\mathcal{T} \subset \mathcal{V}_1 \times \mathcal{V}_2$  such that*

$$|\mathcal{T}| \geq \frac{\varepsilon}{1 + \varepsilon} V_1 V_2, \tag{20a}$$

$$|s(\mathcal{T})| \leq 2^{(1+\varepsilon)(2+H_S(\mathbf{y}))}. \tag{20b}$$



The proof of Lemma 8 is presented in Section IV-E1. Consider now the event that  $(\mathbf{A}, \mathbf{G}^{\otimes n})$  is  $\delta$ -feasible for the random target function  $\mathbf{A}$  (as introduced in Example 1). As we have seen before, this implies the existence of a  $\delta$ -approximation  $\mathbf{A}_\delta$  of  $\mathbf{A}$  such that  $(\mathbf{A}_\delta, \mathbf{G}^{\otimes n})$  is 0-feasible. Let  $\mathbf{S}$  be the corresponding ordered submatrix of  $\mathbf{G}^{\otimes n}$  specifying the encoders, and let  $\phi$  be the corresponding decoder. For fixed  $\varepsilon > 0$ , let  $\mathcal{T} \subset \mathcal{V}_1 \times \mathcal{V}_2$  be the typical set associated with  $\mathbf{S}$ , as guaranteed by Lemma 8. Since  $\phi$  correctly computes  $a_\delta(\cdot, \cdot)$  for all elements of  $\mathcal{V}_1 \times \mathcal{V}_2$ , it does so in particular for all elements of  $\mathcal{T}$ . More formally,

$$a_\delta(\mathbf{v}_1, \mathbf{v}_2) = \phi(s(\mathbf{v}_1, \mathbf{v}_2))$$

for all  $(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{T}$ .

Fix an ordered submatrix  $\mathbf{S}$  of  $\mathbf{G}^{\otimes n}$  of dimension  $V_1 \times V_2$ . Let  $\mathcal{T}$  be the typical set corresponding to  $\mathbf{S}$ . Consider a random  $\mathbf{A}$ , and let  $\mathcal{E}_\mathbf{S}$  be the event that there exists a  $\delta$ -approximation  $\mathbf{A}_\delta$  of dimension  $V_1 \times V_2$  and a mapping  $\phi : s(\mathcal{T}) \rightarrow \mathcal{W}$  such that

$$a_\delta(\mathbf{v}_1, \mathbf{v}_2) = \phi(s(\mathbf{v}_1, \mathbf{v}_2))$$

for all  $(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{T}$ . From the discussion in the last paragraph, we have

$$\mathbb{P}((\mathbf{A}, \mathbf{G}^{\otimes n}) \text{ is } \delta\text{-feasible}) \leq \mathbb{P}(\cup_{\mathbf{S}} \mathcal{E}_\mathbf{S}) \leq \sum_{\mathbf{S}} \mathbb{P}(\mathcal{E}_\mathbf{S}). \quad (21)$$

We continue by upper bounding the probability of the event  $\mathcal{E}_\mathbf{S}$ . Fix a mapping  $\phi$  and let  $\mathcal{A}_\delta^\phi$  denote the set of distinct  $V_1 \times V_2$  matrices  $\mathbf{A}_\delta$  with entries in  $\mathcal{W}$  such that

$$a_\delta(\mathbf{v}_1, \mathbf{v}_2) = \phi(s(\mathbf{v}_1, \mathbf{v}_2))$$

for all  $(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{T}$ . Using the union bound, we have

$$\mathbb{P}(\mathcal{E}_\mathbf{S}) \leq \sum_{\phi} \sum_{\mathbf{A}_\delta \in \mathcal{A}_\delta^\phi} \mathbb{P}(\mathbf{A}_\delta \text{ is a } \delta\text{-approximation of } \mathbf{A}). \quad (22)$$

The number of matrices in  $\mathcal{A}_\delta^\phi$  is at most

$$|\mathcal{A}_\delta^\phi| \leq W^{V_1 V_2 - |\mathcal{T}|} \leq W^{V_1 V_2 / (1+\varepsilon)} \leq W^{U^2 / (1+\varepsilon)},$$

where the second inequality follows from (20a) in Lemma 8. Since there are at most  $W^{|\mathcal{T}|}$  mappings  $\phi$  from  $s(\mathcal{T})$  to  $\mathcal{W}$ , we have

$$\sum_{\phi} |\mathcal{A}_\delta^\phi| \leq W^{|\mathcal{T}|} W^{U^2 / (1+\varepsilon)} \leq \exp(\ln(W) 2^{(1+\varepsilon)(2+H_{\mathbf{S}}(\mathbf{y}))} + \ln(W) U^2 / (1+\varepsilon)) \quad (23)$$

where the last inequality follows from (20b) in Lemma 8.

Consider then a fixed matrix  $\mathbf{A}_\delta$ . The next lemma upper bounds the probability that this fixed  $\mathbf{A}_\delta$  is, in fact, a  $\delta$ -approximation of the random target function  $\mathbf{A}$ .

**Lemma 9.** Fix  $0 < \delta < 1 - 1/W$  and an arbitrary matrix  $\mathbf{A}_\delta$  of dimension  $V_1 \times V_2$  with  $V_1, V_2 \leq U$  and range of cardinality  $W$ . Let  $\mathbf{A}$  be the random target function of dimension  $U \times U$  and range of cardinality  $W$ . Then

$$\mathbb{P}(\mathbf{A}_\delta \text{ is a } \delta\text{-approximation of } \mathbf{A}) \leq \exp(2U \ln(U) - \alpha U^2),$$

with

$$\alpha \triangleq (1 - \delta) \ln(W(1 - \delta)) - (1 - \delta).$$

The proof of Lemma 9 is presented in Section IV-E2. Combining (22), (23), and Lemma 9 shows that for any  $\mathbf{S}$ ,

$$\mathbb{P}(\mathcal{E}_\mathbf{S}) \leq \exp\left(\ln(W) 2^{(1+\varepsilon)(2+H_{\mathbf{S}}(\mathbf{y}))} + 2U \ln(U) - (\alpha - \ln(W)/(1+\varepsilon)) U^2\right).$$

Substituting the above into (21), we have

$$\begin{aligned}
& \mathbb{P}((\mathbf{A}, \mathbf{G}^{\otimes n}) \text{ is } \delta\text{-feasible}) \\
& \leq \sum_{\mathbf{S}} \mathbb{P}(\mathcal{E}_{\mathbf{S}}) \\
& \leq \sum_{\mathbf{S}} \exp\left(\ln(W)2^{(1+\varepsilon)(2+H_{\mathbf{S}}(\mathbf{y}))} + 2U \ln(U) - (\alpha - \ln(W)/(1+\varepsilon))U^2\right) \\
& \leq \exp\left(2nU \ln(X) + \ln(W)2^{(1+\varepsilon)(2+\max_{\mathbf{S}} H_{\mathbf{S}}(\mathbf{y}))} + 2(U+1) \ln(U) - (\alpha - \ln(W)/(1+\varepsilon))U^2\right) \quad (24)
\end{aligned}$$

where the last inequality follows by noting that there are at most  $U^2 X^{2nU}$  ordered submatrices  $\mathbf{S}$  of  $\mathbf{G}^{\otimes n}$  of dimension at most  $U \times U$ .

Now, set

$$\varepsilon \triangleq \frac{1}{\frac{1}{2} \ln(W) - 1},$$

and note that  $\varepsilon \rightarrow 0$  as  $U \rightarrow \infty$  since  $W \geq \omega(1)$  as  $U \rightarrow \infty$ . Recall that

$$\delta \leq 1/(2 \ln(W))$$

by assumption. This implies that

$$\begin{aligned}
\alpha - \frac{\ln(W)}{1+\varepsilon} &= (1-\delta) \ln(W(1-\delta)) - (1-\delta) - \ln(W) + 2 \\
&\geq (1-\delta) \ln(1-\delta) + \delta + 1/2 \\
&\geq 1/2.
\end{aligned}$$

Hence, the right-hand side of (24) converges to zero as  $U \rightarrow \infty$  if the following two conditions hold,

$$\begin{aligned}
n &\dot{<} U, \\
\ln(W)2^{(1+\varepsilon)(2+\max_{\mathbf{S}} H_{\mathbf{S}}(\mathbf{y}))} &\dot{<} U^2.
\end{aligned}$$

In particular, since  $W \leq U^2$  without loss of generality, and since  $\varepsilon \rightarrow 0$  as  $U \rightarrow \infty$ , this is the case whenever

$$\begin{aligned}
n &\dot{<} U, \\
2^{\max_{\mathbf{S}} H_{\mathbf{S}}(\mathbf{y})} &\dot{<} U^2.
\end{aligned}$$

Thus, if

$$\lim_{U \rightarrow \infty} \mathbb{P}((\mathbf{A}, \mathbf{G}^{\otimes n}) \text{ is } \delta\text{-feasible}) > 0,$$

then either

$$n \dot{\geq} U, \quad (25a)$$

or there must exist a submatrix  $\mathbf{S}$  of  $\mathbf{G}^{\otimes n}$  of dimension at most  $U \times U$  such that

$$U^2 \dot{\leq} 2^{H_{\mathbf{S}}(\mathbf{y})}. \quad (25b)$$

Assume that the latter is true. Let  $\mathbf{v}_1, \mathbf{v}_2$  denote independent random variables corresponding to the channel inputs of the two users, as specified by the submatrix  $\mathbf{S}$ . Then we have

$$\begin{aligned}
U^2 &\dot{\leq} 2^{H_{\mathbf{S}}(\mathbf{y})} \\
&\leq 2^{H_{\mathbf{S}}(\mathbf{y}, \mathbf{v}_1)} \\
&= 2^{H_{\mathbf{S}}(\mathbf{v}_1) + H_{\mathbf{S}}(\mathbf{y}|\mathbf{v}_1)} \\
&\leq 2^{\log U + H_{\mathbf{S}}(\mathbf{y}|\mathbf{v}_1)},
\end{aligned}$$

which implies that

$$U \stackrel{\cdot}{\leq} 2^{H_S(\mathbf{y}|\mathbf{v}_1)}. \quad (26)$$

Similarly, we have

$$U \stackrel{\cdot}{\leq} 2^{H_S(\mathbf{y}|\mathbf{v}_2)}. \quad (27)$$

From (25b), (26), (27), it follows that there exists a joint distribution on  $\mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}^n$  of the form

$$p(\mathbf{v}_1, \mathbf{v}_2, \mathbf{y}) = p(\mathbf{v}_1) \times p(\mathbf{v}_2) \times \prod_{i=1}^n p(y[i]|v_1[i], v_2[i])$$

which satisfies

$$\begin{aligned} U^2 &\stackrel{\cdot}{\leq} 2^{H(\mathbf{y})} \leq 2^{\sum_{i=1}^n H(y[i])}, \\ U &\stackrel{\cdot}{\leq} 2^{H(\mathbf{y}|\mathbf{v}_1)} \leq 2^{\sum_{i=1}^n H(y[i]|\mathbf{v}_1[i])}, \\ U &\stackrel{\cdot}{\leq} 2^{H(\mathbf{y}|\mathbf{v}_2)} \leq 2^{\sum_{i=1}^n H(y[i]|\mathbf{v}_2[i])}. \end{aligned}$$

We can then single-letterize the right-hand side of the above inequalities in the usual way, see for example the proof of [18, Theorem 14.3.3]. Then it follows that there exists a joint distribution of the form  $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$  such that

$$\begin{aligned} U^2 &\stackrel{\cdot}{\leq} 2^{n(U)H(y|q)}, \\ U &\stackrel{\cdot}{\leq} 2^{n(U)H(y|x_1, q)}, \\ U &\stackrel{\cdot}{\leq} 2^{n(U)H(y|x_2, q)}. \end{aligned}$$

Thus, if (25b) holds, then the above inequalities provide a list of necessary conditions for  $(\mathbf{A}, \mathbf{G}^{\otimes n})$  to be  $\delta$ -feasible with positive probability. It follows easily that the conditions above are also implied if the alternate condition in (25a) holds, thus concluding the proof. ■

It remains to prove Lemmas 8 and 9.

1) *Proof of Lemma 8:* Consider a variable-length binary Huffman code for the random variable  $\mathbf{y}$  distributed according to (19), and let  $\ell(\mathbf{y})$  be the length of the codeword associated with  $\mathbf{y}$ . By [18, Theorem 5.4.1], the expected length

$$L \triangleq \mathbb{E}(\ell(\mathbf{y}))$$

of the code satisfies

$$H_S(\mathbf{y}) \leq L \leq H_S(\mathbf{y}) + 1. \quad (28)$$

Let  $\mathcal{C} \subset s(\mathcal{V}_1 \times \mathcal{V}_2)$  denote the set of  $\mathbf{y}$  such that  $\ell(\mathbf{y}) \leq (1 + \varepsilon)L$  for some  $\varepsilon > 0$ , and define

$$\mathcal{T} \triangleq s^{-1}(\mathcal{C})$$

as the elements in  $\mathcal{V}_1 \times \mathcal{V}_2$  that are mapped into  $\mathcal{C}$ . We have

$$\begin{aligned} |s(\mathcal{T})| &= |\mathcal{C}| \\ &\stackrel{(a)}{\leq} 2^{(1+\varepsilon)L+1} \\ &\stackrel{(b)}{\leq} 2^{(1+\varepsilon)(H_S(\mathbf{y})+2)}, \end{aligned}$$

where (a) follows since there are at most  $2^{(1+\varepsilon)L+1}$  binary sequences of length at most  $(1 + \varepsilon)L$  and each of them can correspond to at most one value  $\mathbf{y} \in \mathcal{C}$ , and (b) follows from (28).

On the other hand, we have

$$\begin{aligned}
|\mathcal{T}| &= \left| \{(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2 : s(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}\} \right| \\
&\stackrel{(a)}{=} V_1 V_2 \cdot \mathbb{P}(\mathbf{y} \in \mathcal{C}) = V_1 V_2 \cdot \mathbb{P}(\ell(\mathbf{y}) \leq (1 + \varepsilon)L) \\
&\stackrel{(b)}{\geq} \frac{\varepsilon}{1 + \varepsilon} V_1 V_2,
\end{aligned}$$

where (a) follows from (19) and (b) follows from Markov's inequality. Together, this shows the existence of a set  $\mathcal{T}$  with the desired properties.  $\blacksquare$

2) *Proof of Lemma 9:* Fix an arbitrary function  $a_\delta: \mathcal{V}_1 \times \mathcal{V}_2 \rightarrow \mathcal{W}$  with  $V_1, V_2 \leq U$ , and fix arbitrary maps  $f_1: \mathcal{U}_1 \rightarrow \mathcal{V}_1$  and  $f_2: \mathcal{U}_2 \rightarrow \mathcal{V}_2$ . Denote by  $z_{u_1, u_2}$  the indicator variable of the event

$$\{a(u_1, u_2) = a_\delta(f_1(u_1), f_2(u_2))\}.$$

In words,  $z_{u_1, u_2} = 1$  if the target function  $a(\cdot, \cdot)$  is correctly approximated by  $a_\delta(\cdot, \cdot)$  at  $(u_1, u_2)$ . Since the entries of  $\mathbf{A}$  are uniformly distributed over  $\mathcal{W}$ , we have

$$\mathbb{P}(z_{u_1, u_2} = 1) = 1/W.$$

Since the entries of  $\mathbf{A}$  are independent, the number of message pairs for which the target function  $a(\cdot, \cdot)$  is correctly computed using the approximation function  $a_\delta(\cdot, \cdot)$  is then described by a binomial random variable

$$z \triangleq \sum_{u_1, u_2 \in \mathcal{U}} z_{u_1, u_2}$$

with mean  $U^2/W$ . Thus the probability that for fixed maps  $f_1, f_2$ , the function  $a_\delta(\cdot, \cdot)$  is a  $\delta$ -approximation of the random target function  $a(\cdot, \cdot)$  is given by

$$\begin{aligned}
\mathbb{P}(z \geq (1 - \delta)U^2) &= \mathbb{P}(z \geq (1 + W(1 - \delta) - 1)U^2/W) \\
&\stackrel{(a)}{\leq} \left( \frac{\exp(W(1 - \delta) - 1)}{(W(1 - \delta))^{W(1 - \delta)}} \right)^{U^2/W} \\
&\stackrel{(b)}{\leq} \exp(-\alpha U^2),
\end{aligned}$$

where (a) follows from the Chernoff bound (9) and (b) from the definition of  $\alpha$ .

The number of possible maps  $f_1, f_2$  are at most  $V_1^U$  and  $V_2^U$ , respectively. By the union bound, the probability that the function  $a_\delta(\cdot, \cdot)$  is a  $\delta$ -approximation of the random target function  $a(\cdot, \cdot)$  is then at most

$$V_1^U V_2^U \exp(-\alpha U^2) \leq \exp(2U \ln(U) - \alpha U^2),$$

thus concluding the proof.  $\blacksquare$

## APPENDIX A

### PROOF OF LEMMA 6 IN SECTION IV-B

We prove this result by posing it in the framework of the coupon-collector problem, see, e.g., [19, Chapter 3.6]. In each round, a collector obtains a coupon uniformly at random from a collection of  $Y$  coupons. Let  $z$  denote the number of rounds that are needed until the first time  $X^2 - X + 1$  distinct coupons are obtained. Then the event that  $z$  is at most  $X^2$  is equivalent to the number of distinct entries in the  $X \times X$  random channel matrix  $\mathbf{G}$  being at least  $X^2 - X + 1$ .

Let

$$N \triangleq X^2 - X + 1.$$

For the coupon collector problem, the minimum number of rounds  $z$  needed to collect  $N$  distinct coupons can be written as

$$z = \sum_{i=1}^N z_i, \quad z_i \sim \text{Geom}(p_i), \quad p_i \triangleq 1 - \frac{i-1}{Y}, \quad (29)$$

where the  $z_i$ 's are independent random variables and  $\text{Geom}(p_i)$  represents the geometric distribution with parameter  $p_i$ . Observe that  $p_1 \geq p_2 \geq \dots \geq p_N$ .

From the Chernoff bound (8),

$$\begin{aligned} \mathbb{P}(z > X^2) &\leq \min_{t>0} \exp(-tX^2) \prod_{i=1}^N \mathbb{E}(\exp(tz_i)) \\ &\leq \min_{0 < t < -\ln(1-p_N)} \exp(-tX^2) \prod_{i=1}^N \mathbb{E}(\exp(tz_i)). \end{aligned}$$

We have

$$\mathbb{E}(\exp(tz_i)) = \frac{p_i e^t}{1 - (1 - p_i)e^t}$$

for  $t < -\ln(1 - p_i)$ . Since the right-hand side is decreasing in  $p_i$ , we have

$$\mathbb{E}(\exp(tz_i)) \leq \mathbb{E}(\exp(tz_N))$$

for every  $i \in \{1, 2, \dots, N\}$ . This implies that

$$\mathbb{P}(z > X^2) \leq \min_{0 < t < -\ln(1-p_N)} \exp(-tX^2) \left( \frac{p_N e^t}{1 - (1 - p_N)e^t} \right)^N. \quad (30)$$

Since  $Y \geq e^3 X^3$  by assumption, we have  $-\ln(1 - p_N) > 2$  from (29). Assume that this is the case in the following, and set  $t = 2$  in (30). Then

$$\mathbb{P}(z > X^2) \leq \exp\left(-\left(2X^2 - N \ln\left(\frac{p_N e^2}{1 - (1 - p_N)e^2}\right)\right)\right). \quad (31)$$

Now, since  $Y \geq e^3 X^3$ , we have  $p_N \geq 1 - 1/(e^3 X)$ , and thus

$$\frac{p_N}{1 - (1 - p_N)e^2} \leq \frac{1 - e^{-3}/X}{1 - e^{-1}/X} \leq e^{1/X}. \quad (32)$$

Here, the last inequality follows by setting  $b = 1/X$  in the inequality

$$e^b - be^{b-1} - 1 + be^{-3} \geq 0 \quad \text{for all } b \in [0, 1],$$

which follows from the observation that the left-hand side evaluates to zero at  $b = 0$  and is monotonically increasing for  $b \in [0, 1]$ .

Substituting (32) into (31) and using the definition of  $N$  yields

$$\mathbb{P}(z > X^2) \leq \exp\left(-\left(2X^2 - N(2 + 1/X)\right)\right) \leq \exp(-(X - 2)),$$

thus concluding the proof. ■

## APPENDIX B

### PROOF OF LEMMA 7 IN SECTION IV-C

Consider again the coupon collector problem as in Appendix A, and let  $z$  denote the number of rounds required to collect  $N$  distinct coupons. Then the event that  $z$  is at least  $|a^{-1}(\mathcal{W}_1)|$  is equivalent to  $N_1$  being at most  $N$ . Following the proof of Lemma 6, we have from the Chernoff bound (8) that

$$\begin{aligned} \mathbb{P}(N_1 < N) &= \mathbb{P}(z > |a^{-1}(\mathcal{W}_1)|) \\ &\leq \min_{0 < t < -\ln(1-p_N)} \exp(-t|a^{-1}(\mathcal{W}_1)|) \left( \frac{p_N e^t}{1 - (1-p_N)e^t} \right)^N \\ &\leq \min_{0 < t < -\ln(1-p_N)} \exp(-tcU^2) \left( \frac{p_N e^t}{1 - (1-p_N)e^t} \right)^N, \end{aligned} \quad (33)$$

where the last inequality follows since  $|a^{-1}(\mathcal{W}_1)| \geq cU^2$  by assumption, and with

$$p_N \triangleq 1 - \frac{N-1}{Y}.$$

From the definition of  $N$ , we have  $p_N > 2/3$  so that  $-\ln(1-p_N) > 1$ . Choosing  $t = 1/2$  in (33), and noting that

$$\frac{p_N e^{1/2}}{1 - (1-p_N)e^{1/2}} \leq \frac{2e^{1/2}/3}{1 - e^{1/2}/3} \leq e,$$

we obtain

$$\mathbb{P}(N_1 < N) \leq \exp(-(cU^2/2 - N)) \leq \exp(-cU^2/6),$$

thus proving the lemma. ■

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